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Journal of Number Theory 117 (2006) 387–396

JOURNAL OF Number Theory
www.elsevier.com/locate/jnt

On bounding Hecke–Siegel eigenvalues

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Received 23 July 2004; revised 24 May 2005

Available online 19 September 2005

Communicated by D. Goss

Abstract

We use the action of the Hecke operators $\widetilde{T}_j(p^2)$ $(1 \leqslant j \leqslant n)$ on the Fourier coefficients of Siegel modular forms to bound the eigenvalues of these Hecke operators. This extends work of Duke–Howe–Li and of Kohnen, who provided bounds on the eigenvalues of the operator $T(p)$.

MSC: 11F41

Keywords: Siegel modular forms; Hecke operators

1. Introduction

In the theory of elliptic modular forms, the role of Hecke operators is well understood and fundamental. The role of Hecke operators in the theory of Siegel modular forms is less well understood. For instance, an elliptic modular form that is an eigenfunction of all the Hecke operators (and normalized so that its Fourier coefficient $c(1) = 1$) is completely determined by its eigenvalues and the action of the Hecke operators on Fourier coefficients. As exhibited in [6], the eigenvalues and the action of the Hecke operators on Fourier coefficients of a Siegel modular form are not enough to determine the Fourier coefficients, even when we assume we know the coefficients attached to all maximal integral lattices (of which there are infinitely many).

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0022-314X/\$ - see front matter
doi:10.1016/j.jnt.2005.06.013

For each prime p, there are $n+1$ generators of the local Hecke algebra acting on degree n Siegel modular forms, namely $T(p)$, $T_j(p^2)$, $1 \leqslant j \leqslant n$ (defined in the next section). In [3], the authors used representation theory to obtain a bound on the eigenvalues of $T(p)$, p prime, $p \to \infty$. In [7] Kohnen used counting arguments to give a proof of this result, relying on Maass' description of the action of $T(p)$ on Fourier coefficients [10]. Kohnen also extended the result of [3] by considering bounds stronger than Hecke's trivial bound on the growth of Fourier coefficients of Siegel modular forms. As Kohnen comments in his paper, the bound he obtains is not as strong as that conjectured by Satake [12] and Kurokawa [8] (see also [9]).

In this paper we use counting arguments and the description in [5] of the action of "averaged" Hecke operators $\widetilde{T}_j(p^2)$ $(1 \leqslant j \leqslant n)$ on Fourier coefficients to bound the eigenvalues of $\widetilde{T}_j(p^2)$. As discussed below, the operators $\widetilde{T}_j(p^2)$ generate the same algebra as the $T_j(p^2)$, $1 \leqslant j \leqslant n$. As we also discuss below, we can associate the Fourier coefficients of a modular form with integral lattices Λ, which allows us to more easily organize information in our counting arguments. We prove the following theorem.

Theorem. *Let F be a nonsingular Siegel modular form of degree n so that for some j, $1 \leqslant j \leqslant n$, $F\big|\widetilde{T}_j(p^2) = \lambda_j(p^2)$ for almost all primes p. Also suppose that the Fourier coefficients $c(\Lambda)$ of F satisfy the bound $|c(\Lambda)| \ll_F (\operatorname{disc}\Lambda)^{k/2-\gamma}$. Then*

$$|\lambda_j(p^2)| \ll_F p^M \ (\text{as } p \to \infty),$$

where

$$M = \tfrac{1}{4}(j+n-2\gamma+1)^2 + \tfrac{1}{6}(j-n+2\gamma-\tfrac{1}{2})^2 + j(k-j-1).$$

When $\gamma = 0$ we can take $M = \frac{1}{4}(j+n+1)^2 + j(k-j-1)$.

Remark. Our proof actually shows the following. Say the coefficients of F satisfy the bound of the theorem, and $c(\Lambda)$ is a nonzero coefficient of F. Then for any prime p and integer $j \leqslant n$ such that $p \nmid \operatorname{disc}\Lambda$ and $F|\widetilde{T}_j(p^2) = \lambda_j(p^2)F$,

$$|\lambda_j(p^2)| \ll_{F,\Lambda} \ p^M,$$

where M is as in the theorem.

After proving this theorem, we sketch a proof of [3,7] using the language of lattices. While this is essentially Kohnen's proof, the language of lattices allows us to streamline computations by appealing to elementary quadratic form theory.

The reader is referred to [1,4] for facts about Siegel modular forms, and to [2,11] for facts about quadratic forms and lattices.

2. Definitions

First we define degree n Siegel modular forms, which generalize elliptic modular forms. Here the group of fractional linear transformations we consider is

$$Sp_n(\mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_{2n}(\mathbb{Z}) : \ A\,{}^tB, C\,{}^tD \text{ symmetric}, \ A\,{}^tD - B\,{}^tC = I \right\},$$

and the variable of the modular form τ lies in the Siegel upper half-space $\mathcal{H}^{(n)} = \{X + iY:\ X, Y \in \mathbb{R}^{n,n}$ symmetric, $Y > 0$ (as a quadratic form)$\}$. $Sp_n(\mathbb{Z})$ acts on $\mathcal{H}^{(n)}$ by fractional linear transformation:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \tau = (A\tau + B)(C\tau + D)^{-1}.$$

A Siegel modular form of degree n ($n > 1$) and weight k is a holomorphic function $F : \mathcal{H}^{(n)} \to \mathbb{C}$ that transforms with weight k under $Sp_n(\mathbb{Z})$. This means we can write F as an absolutely convergent series

$$F(\tau) = \sum_T c(T) e\{T\tau\},$$

where T runs over all symmetric, even integral, positive semi-definite $n \times n$ matrices, $e\{*\} = \exp(\pi i \operatorname{Tr}(*))$, and

$$F(\tau) \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right. = F(\tau)$$

for every $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z})$. Here we define

$$F(\tau) \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right. = \det(A\ {}^tD - B\ {}^tC)^{k/2}\ (C\tau + D)^{-k}\ F\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \tau \right).$$

F is called singular if $\det T = 0$ for all T in the support of F.

Since $\begin{pmatrix} G^{-1} & 0 \\ 0 & {}^tG \end{pmatrix} \in Sp_n(\mathbb{Z})$ for all $G \in GL_n(\mathbb{Z})$, we have

$$(\det G)^k\ F(\tau) = \sum_T c({}^tGTG)\ e\{T\tau\}$$

(recall $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$). Thus $c(T) = (\det G)^k\ c({}^tGTG)$, so we can rewrite F as a Fourier series on (oriented) integral lattices Λ:

$$F(\tau) = \sum_{\operatorname{cls}\Lambda} c(\Lambda) e\{\Lambda, \tau\},$$

where $\operatorname{cls}\Lambda$ varies over all isometry classes of rank n even integral (oriented) lattices, and

$$e\{\Lambda, \tau\} = \sum_G e\{{}^tGTG\tau\},$$

where, relative to some basis, the quadratic form on Λ is given by the matrix T. Here Λ does not need to be oriented when k is even, and then G varies over $O(\Lambda)\backslash GL_n(\mathbb{Z})$. ($O(\Lambda)$ is the orthogonal group of Λ, i.e. the group of integral matrices G that conjugate a matrix representative T of Λ into itself, meaning ${}^tGTG = T$.) When k is odd, we must impose orientations on our lattices to define F as a Fourier series supported on them, and in this case G varies over $O^+(\Lambda)\backslash SL_n(\mathbb{Z})$, where $O^+(\Lambda) = O(\Lambda) \cap SL_n(\mathbb{Z})$.

For p prime, the Hecke operator $T(p)$ is defined by

$$F|T(p) = p^{n(k-n-1)/2} \sum_{\gamma} F|\delta^{-1}\gamma,$$

where

$$\delta = \begin{pmatrix} pI_n & \\ & I_n \end{pmatrix},$$

$\Gamma = Sp_n(\mathbb{Z})$, $\Gamma' = \delta\Gamma\delta^{-1}$, and γ runs over a complete set of coset representatives for $(\Gamma \cap \Gamma')\backslash\Gamma$. Using the terminology of lattices, Maass' result states that the Λth coefficient of $F|T(p)$ is

$$\sum_{p\Lambda \subseteq \Omega\Lambda} p^{E(\Lambda,\Omega)}\ c(\Omega^{1/p}),$$

where $E(\Lambda, \Omega) = m(1)k + m(p)(m(p)+1)/2 - n(n+1)/2$, $m(a) = \mathrm{mult}_{\{\Lambda:\Omega\}}(a)$, and $\Omega^{1/p}$ denotes the lattice Ω whose quadratic form has been scaled by $1/p$.

The Hecke operators $T_j(p^2)$ $(1 \leqslant j \leqslant n)$ are defined by

$$F|T_j(p^2) = \sum_{\gamma} F|\delta^{-1}\gamma,$$

where

$$\delta = \begin{pmatrix} pI_j & & & \\ & I_{n-j} & & \\ & & \frac{1}{p}I_j & \\ & & & I_{n-j} \end{pmatrix},$$

$\Gamma' = \delta\Gamma\delta^{-1}$, and γ runs over a complete set of coset representatives for $(\Gamma \cap \Gamma')\backslash\Gamma$. In [5] we obtain an explicit set of coset representatives for the above quotient group. When evaluating the action of $T_j(p^2)$ on Fourier coefficients, we encounter incomplete character sums. We complete these character sums by replacing $T_j(p^2)$ with

$$\widetilde{T}_j(p^2) = p^{j(k-n-1)} \sum_{0 \leqslant t \leqslant j} \beta(n-t, j-t)\ T_t(p^2),$$

where $\beta(m, \ell) = \prod_{i=1}^{\ell} \frac{(p^{m-\ell+i}-1)}{(p^i-1)}$ is the number of ℓ-dimensional subspaces of a dimension m space over $\mathbb{Z}/p\mathbb{Z}$. As shown in Theorem 4.1 of [5], the Λth coefficient of $F|\widetilde{T}_j(p^2)$ is

$$\sum_{p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda} p^{E_j(\Lambda,\Omega)}\ \alpha_j(\Lambda, \Omega)\ c(\Omega);$$

here

$$\begin{aligned} E_j(\Lambda, \Omega) = {} & k(m(1/p) - m(p)) + m(p)(m(p) + m(1) + 1) \\ & + (m(1) - n + j)(m(1) - n + j + 1)/2 + j(k - n - 1), \end{aligned}$$

and $\alpha_j(\Lambda,\Omega)$ denotes the number of totally isotropic, codimension $n-j$ subspaces of $(\Lambda\cap\Omega)/p(\Lambda+\Omega)$.

Note that the algebra generated by the $\widetilde{T}_j(p^2)$ is that generated by the $T_j(p^2)$. Thus, since the space of Siegel modular forms has a basis of simultaneous eigenforms for the operators $\{T(p), T_j(p^2): 1\leqslant j\leqslant n,\ p \text{ prime}\}$, the space has a basis consisting of eigenforms for the $\widetilde{T}_j(p^2)$.

3. Proof of theorem

We prove our theorem by counting the Ω in the above sum describing the Λth coefficient of $F|\widetilde{T}_j(p^2)$, and by using known and conjectural bounds on Fourier coefficients.

Choose Λ such that $\operatorname{disc}\Lambda\neq 0$ and $c(\Lambda)\neq 0$. (Since F is not singular, such Λ exists.) Let Q denote the quadratic form on Λ and B the associated symmetric bilinear form so that $Q(x)=B(x,x)$. Let p be any prime with $p\nmid\operatorname{disc}\Lambda$. Since $F|\widetilde{T}_j(p^2)=\lambda_j(p^2)F$, Theorem 4.1 of [5] gives us

$$\lambda_j(p^2)\ c(\Lambda)=\sum_{\Omega} p^{E_j(\Lambda,\Omega)}\ \alpha_j(\Lambda,\Omega)\ c(\Omega),$$

where Ω varies subject to $p\Lambda\subseteq\Omega\subseteq\frac{1}{p}\Lambda$. Since $c(\Omega)=0$ when Ω is not integral, we only consider integral Ω in the above sum.

First we partition this sum according to the invariant factors $\{\Lambda:\Omega\}$; note that the number of choices for $\{\Lambda:\Omega\}$ is completely determined by n. Next, for prescribed $\{\Lambda:\Omega\}$, we rewrite the sum as

$$\sum_{\Omega,R} p^{E_j(\Lambda,\Omega)}\ c(\Omega),$$

where Ω varies over all lattices with $\{\Lambda:\Omega\}$ as prescribed and R varies over all totally isotropic, codimension $n-j$ subspaces of $(\Lambda\cap\Omega)/p(\Lambda+\Omega)$ (recall that $\alpha_j(\Lambda,\Omega)$ counts how many such R exist). Our assumed bound on $c(\Omega)$ can be rewritten in terms of $\operatorname{disc}\Lambda$ and $\{\Lambda:\Omega\}$, as can $E_j(\Lambda,\Omega)$. Thus by counting the pairs (Ω,R) for each choice of $\{\Lambda:\Omega\}$, we produce a bound on $\lambda_j(p^2)$.

We now fix $\{\Lambda:\Omega\}$ and construct all pairs (Ω,R) subject to the above constraints. We will make use of the lemma stated and proved in the following section.

Given Ω such that $p\Lambda\subseteq\Omega\subseteq\frac{1}{p}\Lambda$, the Invariant Factor Theorem [11, p. 214] tells us

$$\Lambda=\Lambda_0\oplus\Lambda_1\oplus\Lambda_2,$$
$$\Omega=p\Lambda_0\oplus\Lambda_1\oplus\frac{1}{p}\Lambda_2,$$

where $\operatorname{rank}\Lambda_0$ is the multiplicity of p in $\{\Lambda:\Omega\}$ and $\operatorname{rank}\Lambda_2$ is the multiplicity of $\frac{1}{p}$ in $\{\Lambda:\Omega\}$. Note that since we require Ω be integral, we must have $Q(\Lambda_2)\equiv 0\pmod{p^2}$ and $B(\Lambda_1,\Lambda_2)\equiv 0\pmod{p}$. Also,

$$(\Lambda\cap\Omega)/p(\Lambda+\Omega)\simeq\Lambda_1/p\Lambda_1.$$

Since $\alpha_j(\Lambda,\Omega)$ counts codimension $n-j$ totally isotropic subspaces of $\Lambda_1/p\Lambda_1$, we must have $\alpha_j(\Lambda,\Omega)=0$ if $\operatorname{rank}(\Lambda_0\oplus\Lambda_2)>j$. Now fix d_0,d_1,d_2 such that $d_0+d_2\leqslant j$, $d_0+d_1+d_2=n$. We will construct and count all pairs (Ω, R) where $d_i=\operatorname{rank}\Lambda_i$ and R is a totally isotropic, codimension $n-j$ subspace of $\Lambda_1/p\Lambda_1$.

Choose $\overline{\Delta_2}$ to be a totally isotropic subspace of $\Lambda/p\Lambda$ with dimension d_2. Since $p\nmid\operatorname{disc}\Lambda$, $\Lambda/p\Lambda$ is a regular space over $\mathbb{Z}/p\mathbb{Z}$. So as shown in our bounding lemma below, the number of choices for $\overline{\Delta_2}$ is bounded by $4^{d_2}p^{d_2(n-d_2)-d_2(d_2+1)/2}$.

Now we extend $\overline{\Delta_2}$ to a totally isotropic space $\overline{\Delta_2}\oplus R$, where $r=\dim R=d_1-n+j=j-d_0-d_2$. Since $\overline{\Delta_2}$ is totally isotropic and $\Lambda/p\Lambda$ is regular, there is a dimension d_2 subspace $\overline{\Delta_2'}$ of $\Lambda/p\Lambda$ so that

$$\overline{\Delta_2}\oplus\overline{\Delta_2'}\simeq\begin{pmatrix}0 & I_{d_2}\\ I_{d_2} & *\end{pmatrix}.$$

Thus $\Lambda/p\Lambda=\left(\overline{\Delta_2}\oplus\overline{\Delta_2'}\right)\perp\overline{J}$ where $\overline{J}$ is a regular space of dimension $n-2d_2$. Hence extending $\overline{\Delta_2}$ to totally isotropic $\overline{\Delta_2}\oplus R$ is equivalent to choosing a totally isotropic dimension r subspace of the regular space $\overline{J}$; by the lemma, this number is bounded by $4^r p^{r(n-2d_2-r)-r(r+1)/2}$.

Now we extend $\overline{\Delta_2}\perp\overline{R}$ to $\overline{\Delta_2}\perp\overline{\Lambda_1}\subseteq\overline{\Delta_2}^{\perp}\subseteq\Lambda/p\Lambda$. We know $\Lambda/p\Lambda=(\overline{\Delta_2}\oplus\overline{\Delta_2}')\perp(R\oplus U)$ for some U. Extending $\overline{\Delta_2}\perp R$ to $\overline{\Delta_2}\perp\Lambda_1$ within $\overline{\Delta_2}^{\perp}$ is equivalent to choosing a dimension d_1-r subspace of U. Since $\dim U=n-2d_2-r$ and $r=j-d_2-d_0$, the lemma shows the number of choices for $\overline{\Lambda_1}$ is bounded by $2^{n-j}p^{(n-j)(d_0-d_2)}$. Also, we now have

$$\Lambda/p\Lambda=\left(\overline{\Delta_2}\oplus\overline{\Delta_2'}\right)\perp\left(\overline{\Lambda_1}\oplus\overline{J'}\right)$$

for some $J'\subseteq\Lambda$.

Next, let Δ be the preimage in Λ of $\overline{\Delta_2}$, and Ω' the preimage in Λ of $\overline{\Delta_2}\perp\overline{\Lambda_1}$. Thus

$$\begin{aligned}\Lambda &= \Lambda_0\oplus\Lambda_1\oplus\Delta_2,\\ \Delta &= p\Lambda_0\oplus p\Lambda_1\oplus\Delta_2,\\ \Omega' &= p\Lambda_0\oplus\Lambda_1\oplus\Delta_2,\end{aligned}$$

where $Q(\Delta_2)\equiv 0\ (\mathrm{mod}\ p)$ and $B(\Lambda_1,\Delta_2)\equiv 0\ (\mathrm{mod}\ p)$. (So here Λ_0, for example, denotes some rank d_0 preimage in Λ of $\overline{\Lambda_0}$.)

Finally, we refine our choice of Δ_2 as follows. In $\Delta/p\Delta$ (scaled by $\frac{1}{p}$), extend $\overline{p\Omega'}=\overline{p\Lambda_1}$ to $\overline{p\Lambda_1}\oplus\overline{\Lambda_2}$ where $\overline{\Lambda_2}$ is totally isotropic of dimension d_2 and independent of $\overline{p\Lambda}$. To count the number of ways we can do this, first recall that

$$\Delta=(\Delta_2\oplus p\Delta_2')\oplus pJ=(\Delta_2\oplus p\Delta_2')\oplus(p\Lambda_1\oplus pJ'),$$

and $\Delta_2\oplus\Delta_2'\simeq\begin{pmatrix}0 & I\\ I & *\end{pmatrix}\ (\mathrm{mod}\ p)$. Thus in $\overline{\Delta}/p\overline{\Delta}$ (scaled by $\frac{1}{p}$), $\overline{\Delta_2}\oplus\overline{p\Delta_2'}$ is a dimension $2d_2$ regular space with $\overline{p\Delta_2'}$ a dimension d_2 totally isotropic subspace; hence $\overline{\Delta_2}\oplus\overline{p\Delta_2'}$

is hyperbolic. So there is a unique $\overline{\Lambda_2'}$ such that $\overline{\Lambda_2'}$ is totally isotropic and $\overline{\Lambda_2'} \oplus \overline{p\Delta_2'} = \overline{\Delta_2} \oplus \overline{p\Delta_2'}$. Here $\overline{pJ}$ is the radical of $\Delta/p\Delta$ (scaled by $\frac{1}{p}$), so isotropic vectors are of the form $x+y$ where x is isotropic in $\overline{\Lambda_2'} \oplus \overline{p\Delta_2'}$ and y is any vector in $\overline{pJ}$. Since we can choose a basis for $\overline{\Lambda_2'}$ such that $\overline{\Lambda_2'} \oplus \overline{p\Delta_2'} \simeq \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, any isotropic vector not in $\overline{p\Lambda}$ must be of the form $x'+y$ where $x' \in \overline{\Lambda_2'}$, $x \neq 0$, and $y \in \overline{pJ}$. Also, a basis for any $\overline{\Lambda_2}$ must project onto a basis for $\overline{\Lambda_2'}$ (else $\overline{\Lambda_2}$ will not be independent of $\overline{p\Lambda}$). So to construct all our extensions, we fix a basis $x_1', \ldots, x_{d_2}'$ for $\overline{\Lambda_2'}$ and let $x_i = x_i' + y_i$ where $y_i \in \overline{pJ'}$. Then we let $\overline{\Lambda_2}$ be the subspace spanned by $x_1, \ldots, x_{d_2}$. Note that $\langle x_1, \ldots, x_{d_2} \rangle \perp \overline{p\Lambda_1} = \langle x_1'', \ldots, x_{d_2}'' \rangle \perp \overline{p\Lambda_1}$ where $x_i'' = x_i + y_i'$, $y_i' \in \overline{p\Lambda_1}$. Thus to construct distinct extensions of $\overline{p\Lambda_1}$, we take $y_i \in \overline{pJ'}$. Thus the number of ways to extend $\overline{p\Lambda_1}$ to $\overline{p\Lambda_1} \oplus \overline{\Lambda_2}$ is the number of ways to choose $y_1, \ldots, y_{d_2} \in \overline{pJ'}$, which is

$$p^{d_2(n-2d_2-d_1)} = p^{d_2(d_0-d_2)}.$$

We let $p\Omega$ be the preimage in Δ of $\overline{p\Lambda_1 \oplus \Lambda_2}$. So

$$\Omega = p\Lambda_0 \oplus \Lambda_1 \oplus \frac{1}{p}\Lambda_2,$$

with $R \subseteq \Lambda_1/p\Lambda_1$. Also, since $Q(\Lambda_2) \equiv 0 \pmod{p^2}$ and $B(\Lambda_1, \Lambda_2) \equiv 0 \pmod{p}$, Ω is integral.

Note that we can construct any pair (Ω, R) through this process. So for fixed d_0, d_1, d_2, we have

$$E_j(\Lambda, \Omega) = k(d_2 - d_0) + d_0(n - d_2 + 1) + r(r+1)/2 + j(k - n - 1),$$

and

$$\#(\Omega, R) \ll p^{d_2(n-d_2)-d_2(d_2+1)/2+r(n-2d_2-r)-r(r+1)/2+(n-j)(d_0-d_2)+d_2(d_0-d_2)}$$

(where $r = j - d_2 - d_0$, and we can take the implied constant to depend only on n). Also, for fixed invariant factors d_0, d_1, d_2,

$$|c(\Omega)| \ll_F (\operatorname{disc} \Omega)^{k/2-\gamma} = p^{2(d_0-d_2)(k/2-\gamma)} \cdot (\operatorname{disc} \Lambda)^{k/2-\gamma},$$

and so (with the d_i still fixed)

$$\left| \sum_{\Omega, R} p^{E_j(\Lambda,\Omega)} c(\Omega) \right| \ll_F p^{E(d_0,d_2)} (\operatorname{disc} \Lambda)^{k/2-\gamma},$$

where $E(d_0, d_2) = -d_0^2 + d_0(j + n + 1 - 2\gamma) - \frac{3}{2}d_2^2 + d_2(j - n - \frac{1}{2} + 2\gamma) + j(k - j - 1)$. (Note that specifying F implicitly specifies n, and the number of terms in the sum is controlled by n.) Thus

$$\left| \lambda_j(p^2) c(\Lambda) \right| \ll_F (\operatorname{disc} \Lambda)^{k/2-\gamma} \sum_{d_0+d_2 \leqslant j} p^{E(d_0,d_2)}.$$

This implies

$$\left|\lambda_j(p^2)\right| \ll_{F,\Lambda} p^M,$$

where M bounds $E(d_0, d_2)$. By calculus (or by completing squares) we find that $E(d_0, d_2)$ is bounded by $M = E\big(\frac{1}{2}(j+n-2\gamma+1), \frac{1}{3}(j-n+2\gamma-\frac{1}{2})\big)$, and keeping in mind we are only concerned with $d_0, d_2 \geqslant 0$, when $\gamma = 0$ we can bound $E(d_0, d_2)$ by $M = E\big(\frac{1}{2}(j+n+1), 0\big)$. This completes the proof of the theorem.

Remark. Here we use the language of lattices to present a proof of the result of [3,7]; this can be interpreted as essentially being the proof given in [7]. Let p, Λ be as above. For $p\Lambda \subseteq \Omega \subseteq \Lambda$, we have $\Lambda = \Lambda_0 \oplus \Lambda_1$, $\Omega = p\Lambda_0 \oplus \Lambda_1$. Let $d_1 = \operatorname{rank} \Lambda_1$. Since $c(\Omega^{1/p}) = 0$ if $\Omega^{1/p}$ is not integral, we only consider those Ω such that $\Omega^{1/p}$ is integral. Thus the Ω we consider are preimages of $\dim d_1$ totally isotropic subspaces of $\Lambda/p\Lambda$. (Since $\Lambda/p\Lambda$ is regular, we have $0 \leqslant d_1 \leqslant n/2$.) For fixed d_1,

$$\#\Omega = \varphi_{d_1}(\Lambda/p\Lambda) \ll p^{d_1(n-d_1)-d_1(d_1+1)/2},$$

$$E(\Lambda, \Omega) = d_1 k - d_1 n + d_1(d_1-1)/2,$$

and

$$\left|c(\Omega^{1/p})\right| \ll p^{(k/2-\gamma)(n-2d_1)}\left|\operatorname{disc} \Lambda\right|^{k/2-\gamma}.$$

Consequently $|\lambda(p)| \ll p^M$, where M is the maximum value of

$$E(d_1) = n(k/2-\gamma) - d_1^2 + d_1(2\gamma-1).$$

So $M = E(\gamma - 1/2) = n(k/2-\gamma) + (\gamma-1/2)^2$. (Note: When $\gamma < 1$, we can take $d_1 = 0$ since this is the nonnegative integer value of d_1 closest to where E attains a maximum.)

4. A bounding lemma

Throughout, let p be a fixed prime.

Lemma. *Let V be a regular space over $\mathbb{Z}/p\mathbb{Z}$ with dimension m. Then the number of dimension d subspaces of V is*

$$\beta(m, d) \leqslant 2^d p^{d(m-d)}.$$

The number of totally isotropic, dimension d subspaces of V is

$$\varphi_d(V) \leqslant 4^d p^{d(m-d)-d(d+1)/2}.$$

Proof. We first bound $\varphi_d(V)$ by constructing all bases for dimension d, totally isotropic subspaces W of V (and then we divide by the number of bases each W has).

First, choose an isotropic (and thus nonzero) vector x_1 in V; as shown in [2, p. 143–146], the number of choices for x_1 is

$$\varphi_1(V) = \begin{cases} p^{m-1} - 1 & \text{if } m \text{ is odd,} \\ (p^{m/2} - 1)(p^{m/2-1} + 1) & \text{if } m \text{ is even and } V \text{ is hyperbolic,} \\ (p^{m/2} + 1)(p^{m/2-1} - 1) & \text{otherwise.} \end{cases}$$

(Note that in Artin's terminology, 0 is considered an isotropic vector.)

Now choose $y_1 \notin x_1^{\perp}$. So the subspace generated by x_1, y_1 is a hyperbolic plane and thus splits V:

$$V = \langle x_1, y_1 \rangle \perp V',$$

where V' is regular of dimension $m-2$, and V' is hyperbolic if and only if V is.

Next we choose an isotropic vector x_2 that is orthogonal to x_1 but not in the span of x_1. Thus $x_2 = ax_1 + x_2'$ where a is some scalar and x_2' is an isotropic vector in V'; so we have $p\varphi_1(V')$ choices for x_2. Now we choose $y_2 \in V'$ so that $y_2 \notin x_2^{\perp}$. Thus the subspace $\langle x_2, y_2 \rangle$ is a hyperbolic plane and orthogonal to $\langle x_1, y_1 \rangle$. So $\langle x_1, y_1 \rangle \perp \langle x_2, y_2 \rangle$ splits V.

Continuing in this fashion, we find that the number of (ordered) bases $x_1, \ldots, x_d$ we can construct for a totally isotropic subspace of V is

$$\begin{cases} \prod_{i=0}^{d-1} p^i (p^{m-2i-1} - 1) & \text{if } m \text{ is odd,} \\ \prod_{i=0}^{d-1} p^i (p^{m/2-i} - 1)(p^{m/2-i-1} + 1) & \text{if } m \text{ is even and } V \text{ is hyperbolic,} \\ \prod_{i=0}^{d-1} p^i (p^{m/2-i} + 1)(p^{m/2-i-1} - 1) & \text{otherwise.} \end{cases}$$

Note that in this way we can construct any dimension d, totally isotropic subspace of V: Say $W = \langle x_1, \ldots, x_d \rangle$ is such a subspace. Then since V is regular, there are $y_1, \ldots, y_d \in V$ so that $\langle x_1, \ldots, x_d, y_1, \ldots, y_d \rangle \simeq \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. By choosing these vectors for $x_1, \ldots, x_d$ and for $y_1, \ldots, y_d$, the preceding algorithm produces W.

The number of (ordered) bases for a given dimension d subspace is

$$\prod_{i=0}^{d-1} p^i (p^{d-i} - 1).$$

Thus

$$\begin{aligned} \varphi_d(V) &\leqslant \prod_{i=0}^{d-1} 2p^{m-i-1-d}/(1 - 1/p) \\ &\leqslant 4^d p^{d(m-d)} \prod_{i=1}^{d} p^{-i} = 4^d p^{d(m-d)-d(d+1)/2}. \end{aligned}$$

Similarly (but more simply), we construct all bases for dimension d subspaces of V by choosing $x_1 \neq 0$, and for $1 \leqslant i < d$, $x_{i+1} \notin \langle x_1, \dots, x_i \rangle$. Thus

$$\begin{aligned}\beta(m,d) &= \prod_{i=0}^{d-1} \frac{p^i(p^{m-i}-1)}{p^i(p^{d-i}-1)} \\ &\leqslant \left(p^{m-d}/(1-1/p)\right)^d = 2^d p^{d(m-d)}. \qquad \square\end{aligned}$$

Acknowledgements

The author thanks Winfried Kohnen for directing her attention to this problem and encouraging her to pursue it, and to Chris Seaton for computer help.

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